

# Tree-colorable maximal planar graphs <sup>1</sup>

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## Abstract

A tree-coloring of a maximal planar graph is a proper vertex 4-coloring such that every bichromatic subgraph, induced by this coloring, is a tree. A maximal planar graph  $G$  is tree-colorable if  $G$  has a tree-coloring. In this article, we prove that a tree-colorable maximal planar graph  $G$  with  $\delta(G) \geq 4$  contains at least four odd-vertices. Moreover, for a tree-colorable maximal planar graph of minimum degree 4 that contains exactly four odd-vertices, we show that the subgraph induced by its four odd-vertices is not a claw and contains no triangles.

*Keywords:* Maximal planar graphs, tree-colorable maximal planar graphs, tree-coloring, claw, triangles.

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## 1. Introduction

The acyclic colorings was first studied by Grünbaum [11], who wrote a long paper to research on the acyclic colorings of planar graphs. He proved

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that every planar graph is acyclic 9-colorable, and conjectured five colors are sufficient. Sure enough, three years later, Borodin [2] (also see [3]) gave a proof of Grünbaum's conjecture by showing that every planar graph is acyclic 5-colorable. In fact, this bound is the best possible for there exist planar graphs with no acyclic 4-colorings [11]. In 1973, Wegner [17] constructed a 4-colorable planar graph  $G$ , each 4-coloring of which possesses a cycle in every bichromatic subgraph. Afterwards Kostochka and Melnikov [12], in 1976, showed that graphs with no acyclic 4-coloring can be found among 3-degenerated bipartite planar graphs.

The research on acyclic 4-colorable planar graphs always aroused more attention. Some sufficient conditions have been obtained for a planar graph to be acyclic 4-colorable. In 1999, Borodin, Kostochka, and Woodall [4] showed that planar graphs under the absence of 3- and 4-cycles are acyclic 4-colorable; In 2006, Montassier, Raspaud, and Wang [15] proved that planar graphs, without 4-, 5-, and 6-cycles, or without 4-, 5-, and 7-cycles, or without 4-, 5-, and intersecting 3-cycles, are acyclic 4-colorable; In 2009, Chen and Raspaud [9] proved that if a planar graph  $G$  has no 4-, 5-, and 8-cycles, then  $G$  is acyclic 4-colorable; Also in 2009, Borodin [5] showed that planar graphs without 4- and 6-cycles are acyclic 4-colorable; Additionally, Borodin in 2011 [6] and 2013 [7] proved that planar graphs without 4- and 5-cycles are acyclic 4-colorable and acyclically 4-choosable, respectively.

## 2. Preliminaries

All of the graphs considered are simple and finite. For a graph  $G$ , we denote by  $V(G)$ ,  $E(G)$ ,  $\delta(G)$  and  $\Delta(G)$  the set of vertices, the set of edges, the *minimum degree* and *maximum degree* of  $G$ , respectively. For a vertex  $u$  of  $G$ ,  $d_G(u)$  is the degree of  $u$  in  $G$ . We call  $u$  a  $k$ -vertex if  $d_G(u) = k$ . If  $k$  is an odd number, we say  $u$  to be an *odd-vertex*, and otherwise an *even-vertex*. If  $d_G(u) > 0$ , then each adjacent vertex of  $u$  is called a *neighbor* of  $u$ . The set of all neighbors of  $u$  in  $G$  is denoted by  $N_G(u)$ . Notice that  $N_G(u)$  does not include  $u$  itself. We then write  $N_G[u] = N_G(u) \cup \{u\}$ . For a subset  $V' \subseteq V(G)$ , denote by  $G[V']$  the subgraph of  $G$  induced by  $V'$ . For more notations and terminologies, we refer the reader to the book [1].

A planar graph  $G$  is called a *plane triangulation* if the addition of any edge to  $G$  results in a nonplanar graph. In this paper, triangulations are also known as *maximal planar graphs*.

A  $k$ -coloring of  $G$  is an assignment of  $k$  colors to  $V(G)$  such that no two adjacent vertices are assigned the same color. Alternatively, a  $k$ -coloring can be viewed as a partition  $\{V_1, V_2, \dots, V_k\}$  of  $V$ , where  $V_i$  denotes the (possibly empty) set of vertices assigned color  $i$ , and is called a *color class* of the coloring.

Let  $f$  be a coloring of a graph  $G$ , and  $H$  be a subgraph of  $G$ . We denote by  $f(H)$  the set of colors assigned to  $V(H)$  under  $f$ . For a cycle  $C$  of  $G$ , if  $|f(C)| = 2$ , then we call  $C$  a *bichromatic cycle* of  $f$ , or say  $f$  *contains bichromatic cycle*  $C$ . An *acyclic  $k$ -coloring* of a graph  $G$  is a  $k$ -coloring with no bichromatic cycles [11].

For a maximal planar graph  $G$ , if  $G$  has an acyclic 4-coloring  $f$ , then not only  $f$  contains no bichromatic cycles, but also any subgraph induced by two color classes of  $f$  is a tree. So, it is more preferable to refer to such an acyclic 4-coloring as a *tree-coloring* of  $G$ . Furthermore, if a maximal planar graph possesses a tree-coloring, then we say this graph is *tree-colorable*.

The *dual* graph  $G^*$  of a plane graph  $G$  is a graph that has a vertex corresponding to each face of  $G$ , and an edge joining two neighboring faces for each edge in  $G$ . It is well-known that the dual graphs of maximal planar graphs are planar cubic 3-connected graphs. Note that  $G$  is a tree-colorable maximal planar graph if and only if its dual graph  $G^*$  contains three Hamilton cycles such that each edges of  $G^*$  is just contained in two of them. Since the problem of deciding whether a planar cubic 3-connected graph contains a Hamilton cycle is NP-complete [10], we can deduce that the problem of deciding whether a maximal planar graph is tree-colorable is NP-complete. In addition, with regard to acyclic 4-colorability of planar graphs, it has been shown that acyclic 4-colorability is NP-complete for planar graphs with maximum degree 5, 6, 7, and 8 respectively and for planar bipartite graphs with the maximum degree 8 [14] [13] [16].

As far as we know, there are no papers that have been written to study the tree-colorability (acyclic 4-colorability) of maximal planar graphs. Because maximal planar graphs contain a large number of 3-, 4-, or 5-cycles, we have reasons to believe that there exist lots of maximal planar graphs without tree-colorings. However, what are the characteristics of a tree-colorable maximal planar graph? In this article, we prove that a tree-colorable maximal planar graph  $G$  with  $\delta(G) \geq 4$  contains at least four odd-vertices. Furthermore, for a tree-colorable maximal planar graph of minimum degree 4 that contains exactly four odd-vertices, we show that the subgraph induced by its four odd-vertices is not a claw and contains no triangles.

### 3. Main results

First, we introduce a novel technique, named *operation of contracting 4-wheel*, which is very useful to the proof of the results throughout this paper.

A  $\ell$ -cycle  $C$  is a cycle of length  $\ell$ . If  $\ell$  is even, we call  $C$  an even cycle, otherwise, an odd cycle. A  $n$ -wheel  $W_n$  (or simply *wheel*  $W$ ) is a graph with  $n+1$  vertices ( $n \geq 3$ ), formed by connecting a single vertex (called the *center* of  $W_n$ ) to all vertices of an  $n$ -cycle.

For a maximal planar graph  $G$  with  $\delta(G) \geq 4$ , it is obvious that any subgraph induced by a vertex and all of its neighbors is a wheel graph. Let  $W$  be a 4-wheel subgraph of  $G$ . The *operation of contracting 4-wheel*  $W$  on  $u, w$  of  $G$ , denoted by  $\mathcal{D}_W^{u,w}(G)$ , is to delete  $v$  from  $G$  and identify vertices  $u$  and  $w$  (replace  $u, w$  by a single vertex  $(u, w)$  incident to all the edges which were incident in  $G$  to either  $u$  or  $w$ ), where  $v$  is the center of  $W$  and  $u, w$  are two nonadjacent neighbors of  $v$ . We denote by  $\zeta_W^{u,w}(G)$  the resulting graph by conducting operation  $\mathcal{D}_W^{u,w}(G)$ . Clearly,

$$\begin{aligned} d_{\zeta_W^{u,w}(G)}((u, w)) &= d_G(u) + d_G(w) - 4, \\ d_{\zeta_W^{u,w}(G)}(x) &= d_G(x) - 2, \\ d_{\zeta_W^{u,w}(G)}(y) &= d_G(y) - 2, \end{aligned} \tag{1}$$

where  $\{x, y\} = N_G(v) \setminus \{u, w\}$ . Notice that  $\zeta_W^{u,w}(G)$  is still a maximal planar graph when  $d_G(x) \geq 5$  and  $d_G(y) \geq 5$ .

We start with a few simple and useful conclusions.

**Lemma 3.1.** *Let  $G$  be a tree-colorable maximal planar graph with a 4-vertex  $v$ . Suppose that  $f$  is a tree-coloring of  $G$ . Then  $|f(N_G(v))| = 3$ , and  $d_G(v_1) \geq 5$ ,  $d_G(v_3) \geq 5$ , where  $v_1, v_3$  are the two nonadjacent neighbors of  $v$  with  $f(v_1) \neq f(v_3)$ .*

*Proof* Let  $v_1, v_2, v_3, v_4$  be the four consecutive neighbors of  $v$  in cyclic order. It naturally follows that  $|f(\{v_1, v_2, v_3, v_4\})| = 3$  for  $f$  is a tree-coloring. Since  $f(v_1) \neq f(v_3)$ , we have  $f(v_2) = f(v_4)$  and  $d_G(v_1) \geq 4$ ,  $d_G(v_3) \geq 4$ . If one of  $v_1, v_3$  is a 4-vertex, say  $v_1$ , then it is unavoidable that  $f$  contains a bichromatic cycle  $v_2v_1v_4v_3v_2$  or  $v_2v_3v_4v_1v_2$ , where  $\{w\} = N_G(v_1) \setminus \{v_2, v, v_4\}$ . So  $d_G(v_1) \geq 5$  and  $d_G(v_3) \geq 5$ .  $\square$

**Lemma 3.2.** *Let  $G$  be a tree-colorable maximal planar graph with a 4-vertex  $v$ , and  $f$  be a tree-coloring of  $G$ . Then  $\zeta_W^{v_1, v_2}(G)$  is still a tree-colorable*

maximal planar graph, where  $W = G[N_G[v]]$  and  $v_1, v_2$  are two nonadjacent neighbors of  $v$  such that  $f(v_1) = f(v_2)$ .

*Proof* By Lemma 3.1  $\delta(\zeta_W^{v_1, v_2}(G)) \geq 3$  which implies  $\zeta_W^{v_1, v_2}(G)$  is still a maximal planar graph. For any  $v \in V(\zeta_W^{v_1, v_2}(G))$ , if  $v \neq (v_1, v_2)$ , let  $f^*(v) = f(v)$ ; otherwise, let  $f^*(v) = f(v_1)$ . Then,  $f^*$  is a tree-coloring of  $\zeta_W^{v_1, v_2}(G)$ .  $\square$

In this paper, we refer to the tree-coloring  $f^*$  of  $\zeta_W^{v_1, v_2}(G)$  in Lemma 3.2 as the *inherited* tree-coloring of  $f$ . Similar to the result of Lemma 3.2, if a tree-colorable maximal planar graph  $G$  contains 3-vertices, then the subgraph of  $G$  obtained by deleting some (or all) 3-vertices is still a tree-colorable maximal planar graph.

Let  $G$  be a graph with a cycle  $C$ . We denote by  $Int(C)$  the subgraph induced by  $V(C)$  and all the vertices in the interior of  $C$ , and denote by  $Ext(C)$  the subgraph induced by  $V(C)$  and vertices in the exterior of  $C$ .

A  $k$ -cycle  $C$  of a connected graph  $G$  is called a separating  $k$ -cycle if the deletion of  $C$  from  $G$  results in a disconnected graph.

**Lemma 3.3.** *A 3-connected maximal planar graph  $G$  is tree-colorable if and only if for any separating 3-cycle  $C$  of  $G$ , both of  $Int(C)$  and  $Ext(C)$  are tree-colorable.*

*Proof* This result is obvious, so we omit the proof.  $\square$

Based on the above tree lemmas, we give the first main result of this section as follow.

**Theorem 3.4.** *A tree-colorable maximal planar graph of minimum degree at least 4 contains at least four odd-vertices.*

*Proof* Let  $G$  be a tree-colorable maximal planar graph with  $\delta(G) \geq 4$ . Then the minimum degree of  $G$  is either 4 or 5. Indeed, it suffices to consider the case of  $\delta(G) = 4$  because  $G$  contains at least twelve 5-vertices by the Euler Formula when  $\delta(G) = 5$ .

If the conclusion fails to hold when  $\delta(G) = 4$ , let  $G'$  be a counterexample on the fewest vertices to the theorem, i.e.  $G'$  is a tree-colorable maximal planar graph of  $\delta(G') = 4$  with  $o(G') < 4$ , where  $o(G')$  is the number of odd-vertices of  $G'$ . It is obvious that  $o(G') = 2$  or  $o(G') = 0$ . Thus, by using the well-known relation

$$\sum_{v \in V(G)} (d(v) - 6) = -12,$$

we can deduce  $G'$  contains at least five 4-vertices.

Let  $f$  be an arbitrary tree-coloring of  $G'$ . If  $G'$  contains no 5-vertices, then for any 4-vertex  $u$  and its two nonadjacent neighbors  $u_1, u_2$  with  $f(u_1) = f(u_2)$ ,  $\zeta_W^{u_1, u_2}(G')$  is still a tree-colorable maximal planar of minimum degree at least 4 and contains at most two odd-vertices by formula (1), where  $W = G'[N_{G'}[u]]$ . This contradicts the assumption of  $G'$ . So we only need to consider the case that  $G'$  contains 5-vertices.

Note that for any 5-vertex  $v$  of  $G'$ , there are at most three 4-vertices in  $N_{G'}(v)$ . Otherwise, if there are four (or five) 4-vertices in  $N_{G'}(v)$ , then  $G'$  is the graph  $G_7$  shown in Figure 1(a). However, it is an easy task to prove that  $G_7$  contains no tree-colorings, and a contradiction. We now turn to show that there are also no three vertices in  $N_{G'}(v)$  with degree 4. If not, let  $v_1, v_2, v_3$  be three 4-vertices of  $N_{G'}(v)$ .

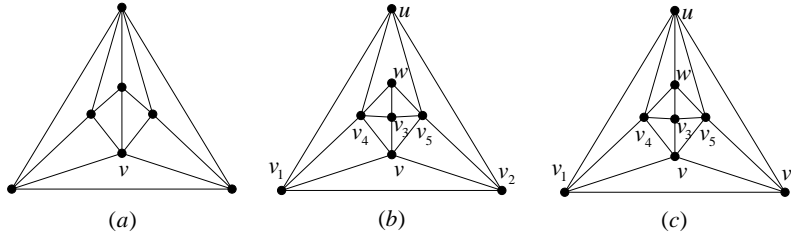


Figure 1: (a) $G_7$ , (b) $H$ , (c) $G_8$

(1)  $v_1, v_2, v_3$  are three consecutive vertices, i.e.  $G'[\{v_1, v_2, v_3\}]$  contains two edges. However, it is readily to check that  $G'$  contains subgraph  $G_7$ , which contradicts the assumption that  $G'$  contains tree-coloring.

(2)  $G'[\{v_1, v_2, v_3\}]$  contains only one edge, w.l.o.g. say  $v_1v_2 \in E(G')$ . Then  $G'$  contains a subgraph  $H$  isomorphic to the graph shown in Figure 1(b). It is easy to see that  $f(v) = f(w)$ .

If  $d'_G(v_4) \geq 6$  and  $d'_G(v_5) \geq 6$ , then  $\zeta_W^{v, w}(G')$  is still a tree-colorable maximal planar graph of minimum degree at least 4 and contains at most two odd-vertices by Lemma 3.2, where  $W = G'[N_{G'}[v]]$ , and a contraction with minimum property of  $G'$ .

If there is a 5-vertex in  $\{v_4, v_5\}$ , say  $v_5$ , then  $G'$  is either the graph  $G_8$  shown in Figure 1(c) that contains four 5-degree vertices (a contradiction with  $G'$ ), or a 3-connected graph with separating 3-cycle  $C = v_4wuv_4$ . For the latter case, either  $\delta(Int(C)) \geq 4$ , or there exists another separating 3-cycle  $C'$  in  $Int(C)$  such that  $\delta(Int(C')) \geq 4$  (because there must be a

separating 3-cycle  $C'$  such that  $\text{Int}(C')$  is 4-connected, otherwise there are 3-vertices in  $\text{Int}(C')$ ). By Lemma 3.3,  $\text{Int}(C)$  ( or  $\text{Int}(C')$ ) is a tree-colorable maximal planar graph with minimum degree at least 4 and contains at most two odd-vertices, which contradicts the assumption of  $G'$ .

The above two cases imply that any 5-vertex  $v$  in  $G'$  has at most two neighbors with degree 4. Since there are at least five 4-vertices in  $G'$ , we can always find a 4-vertex  $v'$  such that  $N_{G'}(v')$  contains no 5-vertices. So, by Lemma 3.1 and 3.2,  $\zeta_W^{v'_1, v'_2}(G')$  is still a tree-colorable maximal planar graph of minimum degree at least 4, where  $W = G'[N_{G'}[v']]$  and  $v'_1, v'_2 \in N_{G'}(v')$  with  $f(v'_1) = f(v'_2)$ . However,  $\zeta_W^{v'_1, v'_2}(G')$  contains at most two odd-vertices, and this contradicts the choice of  $G'$ .  $\square$

By Lemma 3.3, it clearly suffices to consider tree-colorable maximal planar graphs without separating 3-cycle. In what follows, we denote by  $MPG4$  the class of tree-colorable 4-connected maximal planar graphs with exact four odd-vertices. Furthermore, for a graph  $G \in MPG4$ , we denote by  $V^4(G)$  the set of the four odd-vertices of  $G$ . Obviously, the minimum degree of graphs in  $MPG4$  is 4. Now, we turn to discuss the structural properties of graphs in  $MPG4$ .

For a graph  $G$  in  $MPG4$  and a 4-vertex  $v$ , if there are two vertices  $v_1, v_2 \in N_G(v)$  such that  $v_1v_2 \notin E(G)$  and  $\zeta_W^{v_1, v_2}(G)$  is still a graph in  $MPG4$ , then we refer to such vertex  $v$  as a *contractible vertex* of  $G$ .

In order to investigate the structure of the subgraph induced by the four odd-vertices of a graph in  $MPG4$ , we need a lemma as follow.

**Lemma 3.5.** *Let  $G$  be a graph in  $MPG4$ .*

- (1) *If  $G$  contains a 5-vertex  $v$  such that  $N_G(v)$  contains at least three 4-vertices, then either  $G$  is the graph isomorphic to  $G_7$  or  $G_8$ , or  $G$  contains contractible vertices.*
- (2) *If  $G$  contains a 7-vertex  $v$  such that  $N_G(v)$  contains at least five 4-vertices, then  $G$  contains contractible vertices.*
- (3) *If  $G$  contains a 9-vertex  $v$  such that  $N_G(v)$  contains at least six 4-vertices, then either  $G$  has contractible vertices, or  $G$  is the graph isomorphic to Figure 2.*

*Proof* (1). According to the proof of Theorem 3.4, we can know that  $G$  contains either subgraph  $G_7$  or subgraph  $H$ . Since  $G$  is 4-connected, it follows

that either  $G$  is the graph isomorphic to  $G_7$  or  $G_8$ , or  $G$  contains contractible vertices (see the vertex  $v_3$  of graph  $H$  shown in Figure 1(b)).

(2). Let  $v$  be a 7-vertex of  $G$ , and  $N_G(v) = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$  in cyclic order. If  $N_G(v)$  contains at least five 4-vertices, then at least three of them are consecutive, say  $v_1, v_2, v_3$ . Denote by  $v_8$  the common neighbour (except  $v$ ) of them, and then we have  $d_G(v_8) \geq 6$ . Otherwise,  $G$  contains separating 3-cycle  $v_7vv_4v_7$ . By Lemma 3.1 for each tree-coloring  $f$  of  $G$ , we have  $f(v_1) = f(v_3)$ . So  $v_2$  is a contractible vertex.

(3). Let  $v$  be a 9-vertex of  $G$ , and  $N_G(v) = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9\}$  in cyclic order (see Figure 2). If there are three consecutive 4-vertices in  $N_G(v)$ , similarly to (2)  $G$  has contractive vertices. If there are no three consecutive 4-vertices in  $N_G(v)$ , then the number of 4-vertices of  $N_G(v)$  is exactly 6. W.o.l.g. we assume  $v_2, v_3, v_5, v_6, v_8, v_9$  are the six 4-vertices. Because  $G$  is 4-connected, we can assume that the common neighbor(except  $v$ ) of  $v_2$  and  $v_3$  is  $u_1$ , the common neighbor (except  $v$ ) of  $v_5$  and  $v_6$  is  $u_2$ , and the common neighbor (except  $v$ ) of  $v_8$  and  $v_9$  is  $u_3$  (see Figure 2). If one of  $u_1, u_2, u_3$  is a 6-vertices, say  $u_1$ , then  $v_2$  and  $v_3$  are contractible vertices. If  $d_G(u_1) = d_G(u_2) = d_G(u_3) = 5$ , then it follows that  $G$  is the graph isomorphic to Figure 2.  $\square$

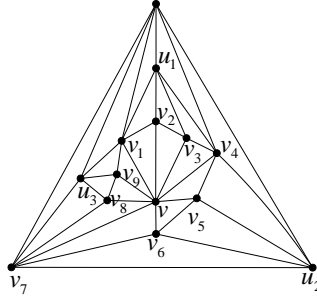


Figure 2: A graph

We then prove that the subgraph induced by the four odd-vertices of a graph in  $MPG4$  contains no triangles.

**Theorem 3.6.** *Let  $G$  be a graph in  $MPG4$  with  $n$  vertices. Then  $G[V^4(G)]$  contains no triangles.*

*Proof* With the help of the software *plantri* developed by *McKay* [8], we



confirm that there are 1,0,2 and 1 graphs in *MPG4* when  $n = 8, 9, 10$  and 11, respectively. We now proceed by induction on  $n$ .

Suppose that the theorem holds for all graphs in *MPG4* with fewer than  $n(\geq 12)$  vertices. Let  $G$  be a graph in *MPG4* with  $n$  vertices, and  $V^4 = \{u_1, u_2, u_3, u_4\}$ . We claim that  $G[V^4(G)]$  contains no triangles. If not, we w.l.o.g. assume  $u_1 u_2 u_3$  is a triangle of  $G[V^4(G)]$ . Then  $G$  contains no contractible vertices. Otherwise let  $u$  be a contractible vertex, i.e. there exist two vertices  $x_1, x_2$  in  $N_G(u)$  such that  $x_1 x_2 \notin E(G)$  and  $\zeta_{W'}^{x_1, x_2}(G) \in \text{MPG4}$ , where  $W' = G[N_G[u]]$ . However it is an easy task to show that the subgraph of  $\zeta_{W'}^{x_1, x_2}(G)$  induced by its four odd-vertices also contains a triangle, and this contradicts the hypothesis.

Notice that  $5 \leq d_G(u_4) \leq 9$ . Otherwise, if  $d_G(u_4) \geq 11$ , then  $G$  contains at least seven 4-vertices. This indicates that there exists a 4-vertex adjacent no 5-vertices by Lemma 3.5 (1). So the 4-vertex is a contractible vertex.

If  $d_G(u_4) = 5$ , then  $G$  contains at least four 4-vertices, and  $N_G(u_4)$  contains at most two 4-vertices by Lemma 3.5 (1); If  $d_G(u_4) = 7$ , then  $G$  contains at least five 4-vertices, and  $N_G(u_4)$  contains at most four 4-vertices by Lemma 3.5 (2); If  $d_G(u_4) = 9$ , then  $G$  contains at least six 4-vertices and  $N_G(u_4)$  contains at most four 4-vertices by Lemma 3.5 (3). So, we can always find a 4-vertex, say  $v'$ , such that  $u_4 \notin N_G(v')$ . Let  $v_1, v_2, v_3, v_4$  be the four consecutive neighbors of  $v'$  (see Figure 3(a)). We now assume  $f(v_1) = f(v_3)$  for any tree-coloring  $f$  of  $G$ , and then  $f(v_2) \neq f(v_4)$  and  $d_G(v_2) \geq 5$ ,  $d_G(v_4) \geq 5$  by Lemma 3.1. In terms of the relation between  $\{u_1, u_2, u_3\}$  and  $N_G(v')$ , there are three cases which can happen. Obviously,  $\{u_1, u_2, u_3\} \not\subset N_G(v')$ .

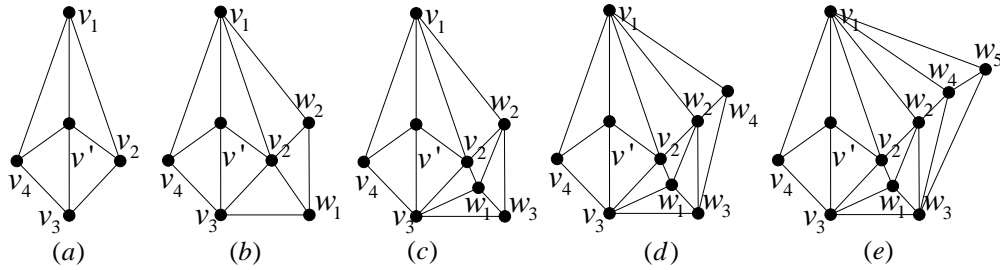


Figure 3:

*Case 1.* One of  $\{u_1, u_2, u_3\}$  belongs to  $N_G(v')$ , say  $u_1$ . By symmetry, it is sufficient to consider  $u_1 = v_1$  or  $u_1 = v_2$ .

If  $u_1 = v_1$ , then  $d_G(v_2) > 5$  and  $d_G(v_4) > 5$ . So  $v'$  is a contractible vertex.

If  $u_1 = v_2$ , it follows  $d_G(u_1) = 5$  (otherwise  $v'$  is a contractible vertex of  $G$ ). Considering that  $N_G(v_2 = u_1) = \{v_1, v', v_3, u_3, u_2\}$ , we have that  $\zeta_{W_1}^{v_1, v_3}(G) - v_2$  is a tree-colorable maximal planar graph with minimum degree at least 4 with only two odd-vertices, where  $W_1 = G[N_G[v']]$ . This contradicts Theorem 3.4.

*Case 2.* Two vertices of  $\{u_1, u_2, u_3\}$  belong to  $N_G(v')$ , w.l.o.g. let  $v_1 = u_1, v_2 = u_2$ . If  $d_G(v_2) \geq 7$ , then  $v'$  is a contractible vertex. If  $d_G(v_2) = 5$ , let  $N_G(v_2) = \{u_1, v', v_3, w_1, w_2\}$ , where  $w_2 = u_3$  (see Figure 3(b)).

*Case 2.1.*  $d_G(w_1) \geq 5$ , then  $\zeta_{W_1}^{v_1, v_3}(G) - v_2$  is still a tree-colorable maximal planar graph with minimum degree 4, but contains at most two odd-vertices, and a contradiction with Theorem 3.4.

*Case 2.2.*  $d_G(w_1) = 4$ , let  $N_G(w_1) = \{v_3, v_2, w_2, w_3\}$  (see Figure 3(c)). Obviously,  $d_G(v_3) \geq 6$  and  $f(w_3) = f(v_2)$ . If  $d_G(w_2) \geq 7$ ,  $w_1$  is a contractible vertex. If  $d_G(w_2) = 5$ , let  $N_G(w_2) = \{v_1, v_2, w_1, w_3, w_4\}$  (see Figure 3(d)). Then  $\zeta_{W_1}^{v_2, w_3}(G) - w_2$  is tree-colorable maximal planar graph with minimum degree at least 4 when  $d_G(w_4) \geq 5$ , but contains at most two odd-vertices. This contradicts to Theorem 3.4; When  $d_G(w_2) = 5$  and  $d_G(w_4) = 4$ , let  $N_G(w_4) = \{v_1, w_2, w_3, w_5\}$  (see Figure 3(e)). Noting that here  $w_4v_4$  is not an edge of  $G$ . Otherwise,  $w_3v_4$  is also an edge of  $G$  for  $d_G(w_4) = 4$ , and  $G$  is a maximal planar graph of order 9 and contains more than four 5-degree vertices). Clearly,  $d_G(v_1) \geq 7$  and  $f(w_5) = f(w_2) = f(v_4)$  or  $f(v')$ , so  $w_5v_3 \notin E(G)$  and  $d_G(w_3) \geq 6$ . This implies that  $w_4$  is contractible vertex.

All of the above discussions show that  $G[V^4(G)]$  contains no triangles.  $\square$

Recall that a star  $S_k$  ( $k \geq 2$ ) is the complete bipartite graph  $K_{1,k}$ , which is a tree with one internal node and  $k$  leaves. A star with 3 edges is called a *claw*, i.e.  $S_3$ . We now in a position to show that the subgraph induced by the four odd-vertices of a graph in  $MPG4$  is not a claw.

**Lemma 3.7.** *Suppose that  $G$  is a 4-connected maximal planar graph satisfying the following three restrictions.*

- 1) *Except one 9-vertex, three 5-vertices, and six 4-vertices, all of other vertices of  $G$  are 6-vertices;*
- 2) *Any two 5-vertices are nonadjacent each other, and all 5-vertices are neighbors of the 9-vertex, and every 5-vertex is adjacent to exactly two 4-vertices;*
- 3) *Each 4-vertex is adjacent to one 5-vertex.*

Then  $G$  is a graph isomorphic to one of the graphs shown in Figure 4(a), (b), (c), (d), (e).

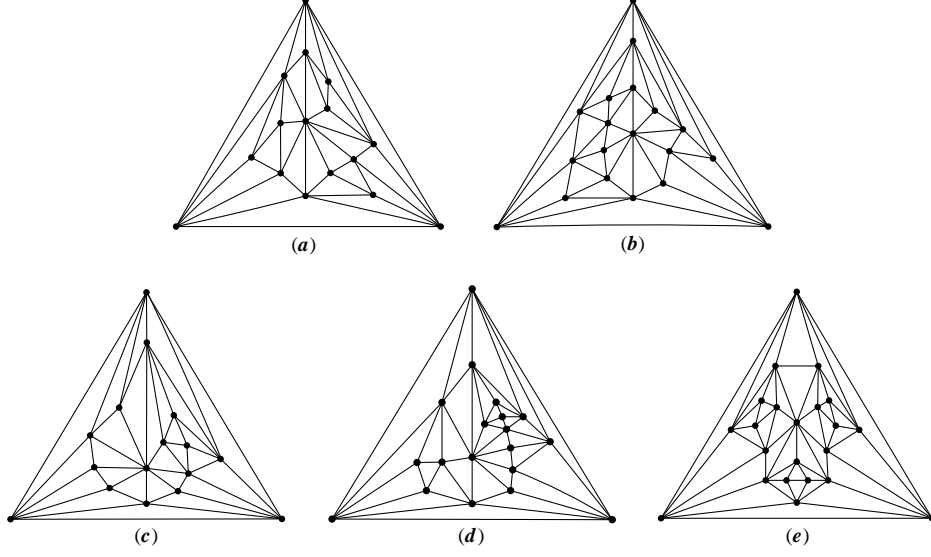


Figure 4: Five unavoidable graphs of Lemma 3.7

*Proof* Let  $u_0$  be the 9-vertex. Since  $G$  is 4-connected,  $G[N_G(u_0)]$  is a cycle  $C$ , denoted by  $C = u_1u_2 \cdots u_9u_1$  (see Figure(a)). We first show that there are no two 5-vertices has a common neighbor on  $C$ . If not, w.l.o.g. we assume  $u_2, u_9$  are two 5-vertices. Clearly,  $u_1$  is a 6-vertex, and  $u_2, u_9$  have no common neighbors (except  $u_0$ ), see Figure 5 (a), where  $v_2, v_3$  (resp.  $v_4, v_5$ ) are the neighbors of  $u_2$  (resp.  $u_9$ ) not on  $C$ , and  $v_1$  is a neighbor of  $u_1$  not on  $C$ . Obviously,  $d_G(v_1) = 6$ . As each 5-vertex has exactly two neighbors of degree 4, we consider the following three cases.

*Case 1.*  $d_G(v_2) = d_G(v_3) = 4$ , i.e.  $v_1v_3, v_1u_3 \in E(G)$ . Then it is impossible  $d_G(v_4) = d_G(v_5) = 4$ , otherwise,  $d_G(v_1) \geq 7$ .

*Case 1.1.*  $d_G(v_4) = d_G(u_8) = 4$ , i.e.  $v_1v_5, v_5u_7 \in E(G)$ , see Figure 5(b). For  $v_1$  is a 6-vertex, we have  $v_5u_3 \in E(G)$ , which implies one of  $u_3$  and  $v_5$  is a vertex of degree at least 7, and a contradiction with  $G$ .

*Case 1.2.*  $d_G(v_5) = d_G(u_8) = 4$ , i.e.  $v_4u_7, v_5u_7 \in E(G)$ , see Figure 5(c). For  $u_7, u_3, v_4, v_1$  are 6-vertices, we can know  $u_5$  is a 3-vertex under the condition  $d_G(u_7) = d_G(u_3) = d_G(v_4) = d_G(v_1) = 6$ , and a contradiction with  $G$ .

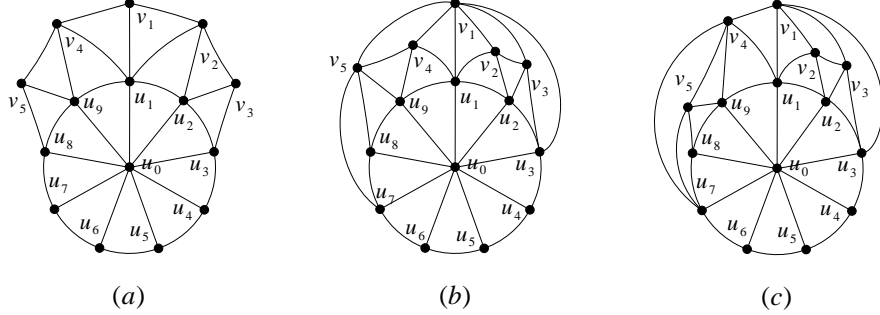


Figure 5: Graphs for Case 1

*Case 2.*  $d_G(v_2) = d_G(u_3) = 4$ , see Figure 6(a).

*Case 2.1.*  $d_G(v_4) = d_G(v_5) = 4$ , then  $v_1v_5, v_1u_8 \in E(G)$ , see Figure 6(a). For  $v_1$  is a 6-vertex, at least one of  $u_8$  and  $v_3$  is a vertex of degree at least 7, and this contradicts to  $G$ .

*Case 2.2.*  $d_G(v_4) = d_G(u_8) = 4$ , then  $v_1v_5, v_5u_7 \in E(G)$ , see Figure 6(b). In this case, for  $v_1, v_3, v_5$  are 6-vertices, and it can be seen that there are no edges between  $v_1$  and  $u_4, u_5, u_6, u_7$  respectively ( By symmetry, we only consider  $v_1u_4 \notin E(G)$  and  $v_1u_5 \notin E(G)$ . If  $v_1u_4 \in E(G)$ ,  $d_G(v_3) = 5$ ; if  $v_1u_5 \in E(G)$ ,  $d_G(u_6) = 3$  based on  $d_G(v_1) = d_G(v_3) = 6$ . We denote the additional neighbor of  $v_1$  by  $v_6$ , see Figure 6(c). By  $d_G(v_3) = d_G(v_5) = d_G(v_6) = 6$  and  $d_G(u_4) = d_G(u_7) = 6$ , we can further known  $d_G(u_5) = d_G(u_6) = 4$ , and a contraction with the condition that there are three 5-vertices on  $C$ .

*Case 2.3.*  $d_G(v_5) = d_G(u_8) = 4$ , then  $v_5u_7, v_4u_7 \in E(G)$  and  $d_G(u_7) = 6$ . Since  $v_1, v_3, v_4$  are 6-vertices, we can know that  $u_7v_1, u_7v_3 \notin E(G)$ . Considering the additional neighbor of  $u_7$ , denote by  $v_6$  (see Figure 6(d)). If  $d_G(v_6) = 4$ ,  $u_6$  will be a 6-vertex since  $v_1, v_3$  are 6-vertices, which implies  $v_6$  is not adjacent to a 5-vertex and a contradiction. So  $v_6$  is a 6-vertex, i.e.  $v_1u_6 \notin E(G)$ . Further, as  $v_3$  is a 6-vertex, there are no edges between  $v_1$  and  $u_4, u_5, u_6$ . Let  $v_7$  be additional neighbor of  $v_3$ . Because  $v_6, u_4, v_7$  are 6-vertices, if  $d_G(u_6) = 5$  we have  $d_G(u_5) = 5$  and a contradiction with  $G$ ; If  $d_G(u_6) = 6$  and  $d_G(u_5) = 5$ , then  $G$  is the graph isomorphic to the graph shown in Figure 6(d), and a contradiction with  $G$ .

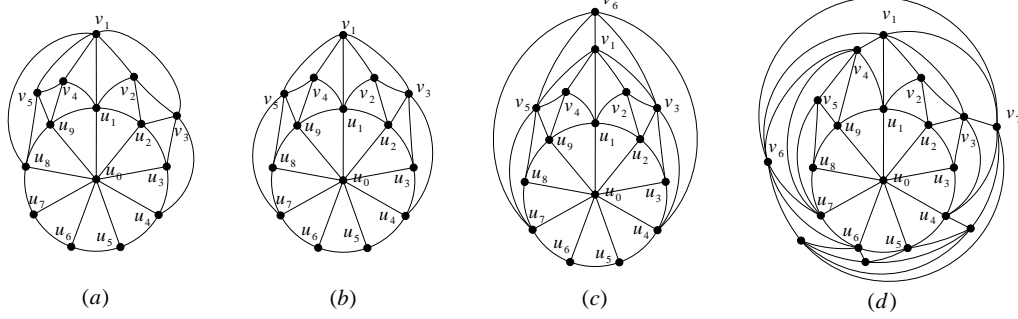


Figure 6: Graphs for Case 2

*Case 3.*  $d_G(v_3) = d_G(u_3) = 4$ . By symmetry, we need only to check the case that  $d_G(v_3) = d_G(u_3) = 4$  and  $d_G(v_5) = d_G(u_8) = 4$ . With the analogously arguing process, it is also to show that this case fails to exist.

Based on the above analysis, we confirm that the three 5-vertices of  $G$  are equably distributed on  $C$ , i.e. no two of them have a common neighbor on  $C$ . In what follows, w.l.o.g. we assume  $u_1, u_4, u_7$  are the three 5-vertices of  $G$  on  $C$ . If there are 4-vertex on  $C$ , suppose w.l.o.g. that  $u_2$  is a 4-vertex. Let  $v_1, v_2$  be additional two neighbors of  $u_1$ , where  $v_1$  is the common neighbor of  $u_1$  and  $u_2$ .

(1)  $d_G(v_1) = 4$ . Since  $d_G(u_3) = 6$  and  $d_G(u_4) = 5$ ,  $u_4$  has additional two neighbors, say  $v_3, v_4$ , where  $v_3$  is the common neighbor of  $u_3$  and  $u_4$  (see Figure 7(a)). Obviously, it is impossible that  $d_G(v_3) = d_G(v_4) = 4$ , otherwise  $v_2$  would be a vertex of degree at least 7. If  $d_G(v_3) = d_G(u_5) = 4$ , then  $d_G(u_7) = 3$  for  $v_2, v_4, u_6, u_9$  are 6-vertices; If  $d_G(v_4) = d_G(u_5) = 4$ , then  $G$  is the graph isomorphic to Figure 4(a).

(2)  $d_G(v_2) = 4$ . We claim that  $v_2u_i \notin E(G)$  for  $i = 3, 4, 5, 6, 7, 8$ . Since  $d_G(v_1) = d_G(u_3) = 6$ , it is indirectly  $v_2u_3, v_2u_4 \notin E(G)$ . If  $v_2u_5 \in E(G)$ , then  $v_1u_5 \in E(G)$  that indicates  $u_5$  is a 6-vertex. Hence we have  $d_G(u_3) = 5$ , a contradiction. Similarly, we have  $v_2u_i \notin E(G)$  for  $i = 6, 7, 8$ . Let  $v_3$  be another neighbor of  $v_2$  and  $v_4$  be the common neighbor of  $v_3$  and  $u_9$  (see Figure 7(b)). Also, there are no edges between  $v_1$  and  $v_4, u_4, u_5, u_6, u_7, u_8$  since  $v_3, u_3$  are 6-vertices, and  $u_i$  is a 6-vertex if  $v_1u_i \in E(G)$  for  $i = 5, 6, 7, 8$ . So, there is another neighbor of  $v_1$ , say  $v_5$  (see Figure 7(b)). Since  $u_3$  and  $u_4$  are 6-vertex and 5-vertex respectively and  $d_G(v_3) = d_G(v_4) = 6$ ,  $u_4$  has additional two neighbors, say  $v_6, v_7$ , where  $v_6$  is the common neighbor of  $u_3$  and  $u_4$  (see Figure 7(b)). If  $v_7, u_5$  are 4-vertices, then  $u_7$  does not contain

two neighbors of degree 4 based on  $d_G(v_3) = d_G(v_5) = d_G(v_6) = 6$ ; If  $v_6, v_7$  are 4-vertices, then one of  $v_3, u_5$  is a vertex of degree at least 7 on the basis of  $d_G(v_5) = 6$ ; If  $v_6, u_5$  are 4-vertices, then  $G$  is isomorphic to the graph shown in Figure 4(b);

(3)  $d_G(u_9) = 4$ , then  $d_G(v_1) = d_G(v_2) = 6$ . Let  $v_3, v_4$  be the other two neighbors of  $u_3$ , obviously  $d_G(v_3) = d_G(v_4) = 6$ , where  $v_3$  is the common neighbor of  $v_1$  and  $u_3$ . So,  $u_4$  is not adjacent to  $v_1, v_2, v_3$  and  $v_4$ . Suppose the additional neighbor of  $u_4$  is  $v_5$ , see Figure 7(c). If  $d_G(v_4) = d_G(u_5) = 4$ , then  $u_7$  does not contain two neighbors of degree 4 based on  $d_G(v_1) = d_G(v_2) = d_G(v_3) = d_G(v_5) = d_G(u_6) = 6$ ; If  $d_G(v_5) = d_G(u_5) = 4$ , then also  $u_7$  does not contain two neighbors of degree 4 based on  $d_G(v_1) = d_G(v_2) = d_G(v_3) = d_G(v_4) = 6$ ; If  $d_G(v_4) = d_G(v_5) = 4$ , then  $G$  is a graph isomorphic to either the graph shown in Figure 4(c) or the graph shown in Figure 4(d).

The above discussions show that when there are 4-vertices on  $C$ ,  $G$  is the graphs isomorphic to the graphs shown in Figure 4(a),(b),(c),(d). Moreover, if there are no 4-vertices on  $C$ , then it is obvious that  $G$  is the graph isomorphic to the Figure 4(e).

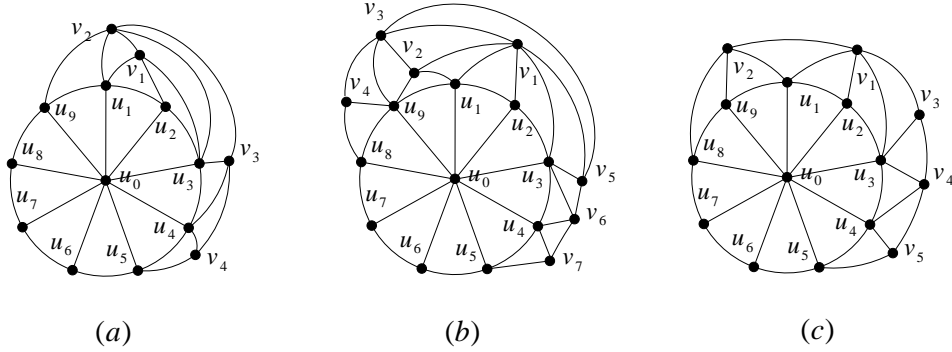


Figure 7: Graphs for Case 3

**Theorem 3.8.** *Let  $G$  be a graph in  $MPG4$ , and  $V^4(G) = \{u_1, u_2, u_3, u_4\}$ . Then  $G[V^4(G)]$  is not a claw.*

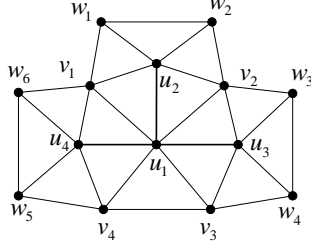


Figure 8: A subgraph

*Proof* If the result fails to hold, select a minimum counterexample  $G'$ , i.e.  $G'$  is a graph in  $MPG4$  with the fewest vertices such that  $G'[\{u_1, u_2, u_3, u_4\}]$  is a claw (see Figure 8(a)). Obviously,  $G'$  does not contain contractible vertices. Thus, it suffices to consider no cases that 5-vertex contains at least three neighbors with degree 4 by Lemma 3.5. Furthermore, according to Theorem 3.6, it follows  $d_{G'}(u_1) \geq 7$ .

If  $d_{G'}(u_1) = 7$ , then  $G'$  contains at least five 4-vertices. So  $G'$  contains a contractible vertex when there is at least one 7-vertex in  $\{u_2, u_3, u_4\}$ ; However, if  $d_{G'}(u_2) = d_{G'}(u_3) = d_{G'}(u_4) = 5$ , (w.l.o.g. see Figure 8), then  $d_{G'}(v_1) \geq 6$  and  $d_{G'}(v_2) \geq 6$ . Indeed, if  $V' = \{w_1, w_2, w_3, w_4, w_5, w_6, v_3, v_4\}$  contains  $\ell (=5, 6, 7)$  4-vertices, then there are at least two vertices  $x, y$  in  $N_G[V']$  such that  $d_G(x) + d_G(y) \geq 2\ell + 4$  by the properties of  $G'$  (All of vertices of  $G'$  except  $u_1, u_2, u_3, u_4$  are even-vertices) in this case. So there are at least one 4-vertex without neighbors of degree 5, which means that  $G'$  contains a contractible vertex and this contradicts the choice of  $G'$ .

If  $d_{G'}(u_1) = 9$ , then there are at least six 4-vertices in  $G'$ . In this case, because each 5-vertex contains at most two neighbors of degree 4, it suffices to consider the unique case:  $G'$  contains exact six 4-vertices,  $d_{G'}(u_2) = d_{G'}(u_3) = d_{G'}(u_4) = 5$ , and all other vertices of  $G'$  have degree 6. Otherwise,  $G'$  contains contractible vertices. Thus, it requires that  $N_G(\{u_2, u_3, u_4\})$  contains six 4-vertices and each of  $u_2, u_3, u_4$  contains exactly two distinct 4-vertices. Then,  $G'$  is one of the graphs shown in Figure 4 by Lemma 3.7, which is not a tree-colorable maximal planar graph.

If  $d_{G'}(u_1) \geq 11$ , then  $G'$  contains at least seven 4-vertices. So at least one 4-vertex has no neighbors of degree 5, and this 4-vertex is a contractible vertex of  $G'$ , and a contradiction.  $\square$

Based on the discussion of Theorem 3.6 and 3.8, we have figured out the impossible structure of the subgraph induced by the four odd-vertices for a

graph in *MPG4*. However, all other structures of this subgraph can appear, including a 4-cycle (see Figure 9 (a)), a path on 4 vertices (see Figure 9 (b)), two vertex-disjoint  $K_2$  (see Figure 9 (c)), a path on 3 vertices and a isolated vertex (see Figure 9 (d)), a  $K_2$  and two isolated vertices (see Figure 9 (e)), and four isolated vertices see Figure 9 (f)).

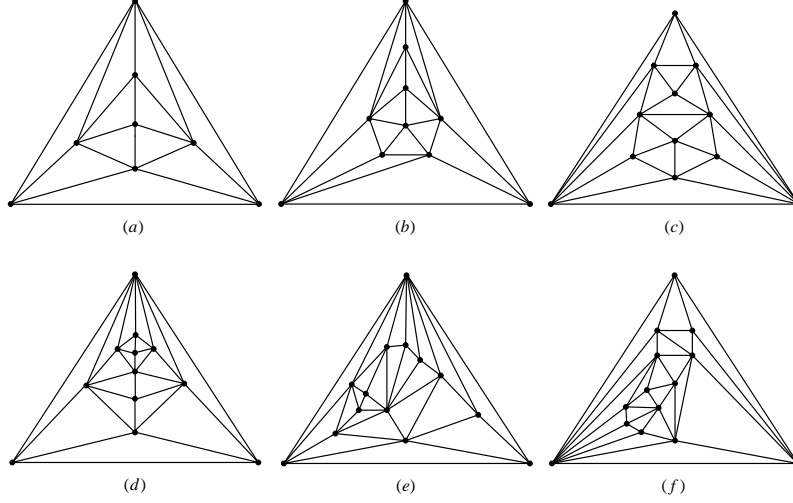


Figure 9: Examples of possible structures of the subgraph induced by the four odd-vertices of a graph in *MPG4*

**Remark.** In this article, we investigated a class of maximal planar graphs, called tree-colorable maximal planar graphs. We proved that a tree-colorable maximal planar graph  $G$  with  $\delta(G) \geq 4$  contains at least four odd-vertices. In addition, for a graph  $G$  in *MPG4*, we showed that the subgraph induced by its four odd-vertices is not a claw and contains no triangles.

With the results we have gained, one can construct maximal planar graphs that contain no tree-colorings. However, for a given maximal planar graph  $G$  that contains exactly four odd-vertices, how to determine whether  $G$  is tree-colorable is still unclear. Exploring the sufficient conditions for  $G$  to be tree-colorable is an challenging task, which we will research on in the future.

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